Photon as the Magnetic Monopole

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Abstract

In this paper we reinvestigate photon. We show that photon can be identified with the Dirac magnetic monopole. We give a model of quantum electrodynamics from which we derive photon as a quantum loop of this model. This nonlinear loop model of photon is exactly solvable and thus may be regarded as a quantum soliton. From the winding numbers of this loop model of photon we derive the quantization property of energy of Planck's formula of radiation and the quantization property of electric charge. We show that these two quantizations are just the same quantization when photon is identified with the magnetic monopole. From this nonlinear model of photon we also construct a model of electron which has a mass mechanism for generating mass to electron.

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1 Introduction

It is well known that the quantum era of physics began with the quantization of energy of electromagnetic field from which Planck derived the radiation formula. Einstein then introduced the light-quantum to explain the photoelectric effects. This light-quantum was later regarded as a particle called photon[1][2][3]. Later quantum mechanics was developed and we came into the modern quantum era. Following the set up of quantum mechanics the quantization of the electromagnetic field and the theory of quantum electrodynamics(QED) was then also set up.

In this development of quantum theory of physics the photon plays a special role. While it is as the beginning of quantum physics it is not as easy to be understood as the quantum mechanics of other particles described by the Schroedinger equation. Indeed Einstein was carefully to regard the light-quantum as a particle and it was much later that the light-quantum was finally accepted as a particle called photon [1]. Then the quantum field theory of electromagnetic field was developed for the photon. However it is well known that this quantum field theory has difficulties such as the ultraviolet divergences. It is because of the difficulty of understanding the photon that Einstein once said "What is the photon?" [1].

On the other hand based on the symmetry of the electric and magnetic field described by the Maxwell equation and based on the complex wave function of quantum mechanics Dirac derived the concept of magnetic monopole which is hypothetically considered as a particle with magnetic charge as analogous to the electron with electric charge [4][5]. An important feature of this magnetic monopole is that it gives the quantization of electric charge. Thus it is interesting and important to find such particles. However despite of many searches it seems that no such particles have been found.

In this paper we shall establish a mathematical model of photon to show that magnetic monopole can be identified with the photon. Before giving the detailed model let us consider some thoughts for the identification of these two particles, as follows.

First we have that photon is regarded as the basic quantum particle of the electromagnetic field while the magnetic monopole is a hypothetical particle derived from the electromagnetic field. Thus if the magnetic monopole is not the photon then we have two kind of quantum particles derived from the electromagnetic field and that the magnetic monopole is derived from the photon because photon is the basic quantum particle of the electromagnetic field. From this we then have the odd conclusion that an elementary quantum particle (i.e. the magnetic monopole) is derived from another elementary quantum particle (i.e. the photon). If we identify these two particles then this odd conclusion can be resolved.

For the identification of these two particles let us reinvestigate the quantum field theory of photon in the literature [6]. It is well known that we have the quantum field theory of the free Maxwell equation which is as the basic quantum theory of photon. Let us consider some points of this theory, as follows. First we have that this free field theory is a linear theory and the model of the quantum particles obtained from this theory is linear. However from the thought of soliton we have that stable particle should be a soliton which is of nonlinear nature. Secondly we have that the quantum particles of this quantum theory of Maxwell equation are collective quantum effects just like the phonons which are elementary exictation of a statistical model. These phonons are usually considered as quasi-particles and are not regarded as real particles. In regarding the Maxwell equation as a statistical wave equation of electromagnetic field we have that the quantum particles of the quantum theory of Maxwell equation are analogous to the phonons. Thus they should be regarded as quasi-photons which have properties of photons but are not a complete description of photons.

In this paper we shall set up a nonlinear model of photon. In this model we show that photon can be identified with the Dirac magnetic monopole. We give a model of quantum electrodynamics from which we derive photon as a quantum loop of this model. This nonlinear loop model of photon is exactly solvable and thus may be regarded as a quantum soliton. From the winding numbers of this loop model of photon we derive the quantization property of energy of Planck's formula of radiation and the quantization property of charge. We show that these two quantizations are just the same quantization when photon is identified with the magnetic monopole. From this nonlinear model of photon we also construct a model of electron which has a mass mechanism for generating mass to electron.

This paper is organized as follows. In section 2 we give a brief description of a quantum gauge model of electrodynamics. With this model in section 3 we introduce the Dirc-Wilson loop. We show that this loop is a nonlinear exactly solvable model and thus can be regarded as a soliton. We identify this Dirc-Wilson loop as the photon when the gauge group is the $U(1)$ group. To investigate the properties of this Dirc-Wilson loop in section 4 we derive a chiral symmetry from the gauge symmetry of this quantum model. From this chiral symmetry in section 5 we derive a conformal field theory which includes the affine Kac-Moody algebra and the Knizhnik-Zamolodchikov (KZ) equation. A main point of our model on the KZ equation is that we can derive two KZ equations which are inversely dual to each other. This duality is the main point for the Dirc-Wilson loop to be exactly solvable and to have a winding property which gives properties of photon. In section 6 from the KZ equations we solve the Dirac-Wilson loop in a form with a winding property. From this winding property of the Dirc-Wilson loop in section 7 we derive the quantization of energy and the quantization of electric charge which are properties of photon and magnetic monopole. We then show that these two quantizations are just the same quantization and we identify photon with the magnetic monopole. From this nonlinear model of photon in section 8 we also give a model of electron. In this model of electron we give a mass mechanism for generating mass to electron.

2 A Quantum Gauge Model

Let us construct a quantum gauge model for photon which is a mathematical model similar to the Wiener measure of the Brownian motion, as follows. In probability theory we have the Wiener measure ν which is a measure on the space $C[t_0, t_1]$ of continuous functions [7]. This probability measure (which may also be regarded as a path integral) is a well defined mathematical theory for the Brownian motion and it may be symbolically written in the following form:

$$d\nu = e^{-L_0} dx \tag{1}$$

where $L_0 := \frac{1}{2}\int_{t_0}^{t_1} \left(\frac{dx}{dt}\right)^2 dt$ is the energy integral of the Brownian particle and $dx = \frac{1}{N}\prod_t dx(t)$ is symbolically a product of Lebesgue measures $dx(t)$ and N is a normalized constant.

Let us then follow this method to construct a measure for a quantum model of electrodynamics, as follows. Similar to the Wiener measure we construct a measure for a quantum model of electrodynamics from the following energy integral:

$$\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}\left(\frac{dA_1}{ds}-\frac{dA_2}{ds}\right)^*\left(\frac{dA_1}{ds}-\frac{dA_2}{ds}\right)+\sum_{i=1}^{2}\left(\frac{dZ}{ds}-ieA_iZ\right)^*\left(\frac{dZ}{ds}-ieA_iZ\right)]ds \tag{2}$$

where s denotes the proper time in relativity and e denotes the electric charge. The complex variable Z will represent an electron and the complex variables A_1 and A_2 represent electromagnetic field from which we shall construct photon. Here we allow A_1 and A_2 to be complex valued.

The integral (2) is invariant under the following gauge transformation:

$$\begin{array}{ll} Z'(s) & := Z(s)e^{iea(s)} \\ Z'^*(s) & := Z(s)^*e^{-iea(s)} \\ A_i^{'}(s) & := A_i(s)+\frac{da}{ds} \\ A_i^{'*}(s) & := A_i^*(s)+\frac{da}{ds} \quad i=1,2 \end{array} \tag{3}$$

where $a(s)$ is a complex valued function. In this gauge transformation the pair variables Z^* and Z are regarded as independent variables. After the gauge transformation they are set to be complex conjugate to each other. The other pair variables Z'^* and Z', A_i^* and A_i, $A_i^{'*}$ and $A_i^{'}$ are similarly treated.

We remark that a feature of (2) is that it is not formulated with the four-dimensional space-time but is formulated with the one dimensional proper time. This one dimensional nature let this measure avoid the usual utraviolet divergence difficulty of quantum fields.

We can generalize this gauge model of photon with $U(1)$ gauge symmetry to nonabelian gauge models. As an illustration let us consider $SU(2)\otimes U(1)$ gauge symmetry where $SU(2)\otimes U(1)$ denotes the direct product of the groups $SU(2)$ and $U(1)$ Similar to (2) we consider the following energy integral:

$$L:=\frac{1}{2}\int_{s_0}^{s_1}[\frac{1}{2}Tr(D_1A_2-D_2A_1)^*(D_1A_2-D_2A_1)+(D_1Z)^*(D_1Z)+(D_2Z)^*(D_2Z)]ds \tag{4}$$

where $Z=(z_1,z_2)^T$ is a two dimensional complex vector; $A_j=\sum_{k=0}^{3}A_j^kt^k$ $(j=1,2)$ where A_j^k denotes a component of a gauge field A^k; $t^k=ieT^k$ denotes a generator of $SU(2)\otimes U(1)$ where T^k denotes a self-adjoint generator of $SU(2)\otimes U(1)$ and e denotes the charge of interaction; and $D_j=\frac{1}{r(s)}\frac{d}{ds}-A_j$, $D_j^*=\frac{1}{r(s)^*}\frac{d}{ds}-A_j^*$, $(j=1,2)$ where $r(s)\neq 0$ is a continuous function on $[s_0,s_1]$.

From (4) we can develop a nonabelian gauge model as similar to that for the above abelian gauge model. We have that (4) is invariant under the following gauge transformation:

$$\begin{array}{ll} Z'(s) & := U(a(s))Z(s) \\ Z'^*(s) & := Z(s)^*U^{-1}(a(s)) \\ A_j'(s) & := \frac{1}{r(s)}U(a(s))A_j(s)U^{-1}(a(s))+U(a(s))\frac{dU^{-1}}{ds}(a(s)), \\ A_j'^*(s) & := \frac{1}{r(s)^*}U(a(s))A_j^*(s)U^{-1}(a(s))+U(a(s))\frac{dU^{-1}}{ds}(a(s)), j=1,2 \end{array} \tag{5}$$

where $U(a(s))=e^{a(s)}$ and $a(s)=\sum_k a^k(s)t^k$. We shall mainly consider the case that a is a function of the form $a(s)=\omega(z(s))$ where ω and z are analytic functions.

We remark that since the above model is a gauge model with a gauge invariance it will be degenerate and we need a gauge fixing to let this model be nondegenerate [8]. A gauge model may have various gauge fixing conditions. As an example we have that the Maxwell equation is a gauge model for electrodynamics. It has various gauge fixing conditions such as the Lorentz gauge condition, the Feynman gauge condition, etc. We shall later adopt a gauge fixing condition for the above gauge model.

3 Dirac-Wilson Loop

Similar to the Wilson loop in quantum field theory from our quantum model we can also introduce an analogue of Wilson loop, as follows.

Definition. A Dirac-Wilson loop (or Wilson loop) $W(C)$ is defined by :

$$W(C) := W(z_0, z_1) := Pe^{\int_C A_j dx^j} \tag{6}$$

where C denotes a continuous closed curve which is of the following form:

$$C(s) = (x^1(z(s)), x^2(z(s))), \qquad s^{'} \leq s \leq s^{''} \tag{7}$$

where $s_0 \leq s^{'} \leq s^{''} \leq s_1$ and $z(\cdot)$ is a continuously differentiable curve in the complex plane such that $z_0 := z(s^{'}) = z(s^{''}) =: z_1$. This closed curve C is in a two dimensional plane (x^1, x^2) with complex coordinates x^1, x^2 which is dual to (A_1, A_2). As usual the notation P in the definition of $W(C)$ denotes a path-ordered product [9][10][11].

For the curve $C(s) = (x^1(z(s)), x^2(z(s)))$ to be nontrivial we suppose that

$$\frac{dx^1(z(s))}{ds} + \frac{dx^2(z(s))}{ds} \neq 0, \qquad s' \leq s \leq s'' \tag{8}$$

Then we define a continuous function $r \neq 0$ on $[s_0, s_1]$ by

$$r(s) := \begin{cases} \frac{dx^1(z(s))}{ds} + \frac{dx^2(z(s))}{ds} & \text{for } s \in [s', s''] \\ \frac{dx^1(z(s'))}{ds} + \frac{dx^2(z(s'))}{ds} & \text{for } s \in [s_0, s'] \cup [s'', s_1] \end{cases} \tag{9}$$

Let us give some remarks on the above definition of Dirac-Wilson loop, as follows.

1) We use the notation $W(z_0, z_1)$ to mean that this Dirac-Wilson loop $W(C)$ is based on the closed curve $z(\cdot)$ in the complex plane which starts at z_0 and ends at z_1 with $z_0 = z_1$. Thus this notation $W(z_0, z_1)$ denotes the Dirac-Wilson loop $W(C)$ constructed from the whole curve $z(\cdot)$. Here for convenience we only use the end points z_0 and z_1 of the curve $z(\cdot)$ to denote this Dirac-Wilson loop.

Then we extend the definition of $W(C)$ to the case that C is not a closed curve with $z_0 \neq z_1$. When C is not a closed loop we shall called $W(z_0, z_1)$ as a Wilson line.

2) We use the above Dirac-Wilson loop $W(C)$ to represent the unknot (Also called the trivial knot). When the gauge group is the $U(1)$ group we shall use it to model the photon.

3) In constructing the Dirac-Wilson loop we need to choose a representation of the $SU(2)$ group. Let us here choose the tensor product of the usual two dimensional representation of the $SU(2)$ for constructing the Dirac-Wilson loop.

4 A Chiral Symmetry

For a given curve $C(s) = (x^1(z(s)), x^2(z(s))), s_2 \leq s \leq s_3$ which may not be a closed curve we define $W(z_0, z_1)$ by (6) where z_0 may not equal to z_1. By following the usual approach of deriving a chiral symmetry from a gauge transformation of a gauge field we have the following chiral symmetry which is derived by applying an analytic gauge transformation with an analytic function ω for the transformation:

$$W(z_0, z_1) \mapsto U(\omega(z_1))W(z_0, z_1)U^{-1}(\omega(z_0)) \tag{10}$$

This chiral symmetry is analogous to the chiral symmetry of the usual guage theory where U denotes an element of the gauge group [10]. Let us derive (10) as follows. Let $U(s) := U(\omega(z(s)))$. Following Kauffman [10] we have

$$\begin{aligned} & U(s+ds)(1 + dx^\mu A_\mu)U^{-1}(s) \\ = \; & U(s+ds)U^{-1}(s) + dx^\mu U(s+\triangle s)A_\mu U^{-1}(s) \\ = \; & 1 + \frac{dU}{ds}(s)U^{-1}(s)ds + dx^\mu U(s+ds)A_\mu U^{-1}(s) \\ \approx \; & 1 + \frac{dU}{ds}(s)U^{-1}(s)ds + dx^\mu U(s)A_\mu U^{-1}(s) \\ =: \; & 1 + \frac{1}{r(s)}\frac{dU}{ds}(s)U^{-1}(s)[(\frac{dx^1}{ds} + \frac{dx^2}{ds})ds] + U(s)[\frac{dx^1}{ds}A_1 + \frac{dx^2}{ds}A_2]U^{-1}(s)ds \\ =: \; & 1 + \sum_{\mu=1}^{2} \frac{dx^\mu}{ds}[\frac{1}{r(s)}\frac{dU}{ds}(s)U^{-1}(s) + U(s)A_\mu U^{-1}(s)]ds \\ =: \; & 1 + dx^\mu A'_\mu \end{aligned} \tag{11}$$

where

$$A'_\mu := \frac{1}{r(s)}\frac{dU}{ds}(s)U^{-1}(s) + U(s)A_\mu U^{-1}(s) \tag{12}$$

From (11) we have that (10) holds.

As analogous to the chiral symmetry of the WZW model in conformal field theory [14] from the above chiral symmetry we have the following formulas for the variations $\delta_\omega W$ and $\delta_{\omega'} W$ with respect to the chiral symmetry:

$$\delta_\omega W(z,z') = W(z,z')\omega(z) \tag{13}$$

and

$$\delta_{\omega'} W(z,z') = -\omega'(z')W(z,z') \tag{14}$$

where z and z' are independent variables and $\omega'(z') = \omega(z)$ when $z' = z$. In (13) the variation is with respect to the z variable while in (14) the variation is with respect to the z' variable. This two-side-variations when $z \neq z'$ can be derived as follows. For the left variation we may let ω be analytic in a neighborhood of z and continuous differentiably extended to a neighborhood of z' such that $\omega(z') = 0$ in this neighborhood of z'. Then from (10) we have that (13) holds. Similarly we may let ω' be analytic in a neighborhood of z' and continuous differentiably extended to a neighborhood of z such that $\omega'(z) = 0$ in this neighborhood of z. Then we have that (14) holds.

5 Affine Kac-Moody Algebra

Now let us derive a quantum loop algebra (or the affine Kac-Moody algebra) structure from the Wilson line $W(z,z')$. To this end let us first consider the classical case that the Wilson line $W(z,z')$ is formed by classical gauge field. Since $W(z,z')$ is constructed from $SU(2)\otimes U(1)$ we have that the mapping $z \to W(z,z')$ (We consider $W(z,z')$ as a function of z with z' being fixed) has a loop group structure [12][13]. For a loop group we have the following generators:

$$J_n^a = t^a z^n \qquad n = 0, \pm 1, \pm 2, ... \tag{15}$$

These generators satisfy the following algebra:

$$[J_m^a, J_n^b] = if_{abc}J_{m+n}^c \tag{16}$$

This is the so called loop algebra [12][13]. Let us then introduce the following generating function J:

$$J(w) = \sum_a J^a(w) = \sum_a j^a(w)t^a \tag{17}$$

where we define

$$J^a(w) = j^a(w)t^a := \sum_{n=-\infty}^{\infty} J_n^a(z)(w-z)^{-n-1} \tag{18}$$

From J we have

$$J_n^a = \frac{1}{2\pi i}\oint_z dw(w-z)^n J^a(w) \tag{19}$$

where $\oint_z$ denotes a closed contour integral with center z. This formula can be interpreted as that J is the generator of the loop group and that J_n^a is the directional generator in the direction $\omega^a(w) = (w-z)^n$. We may generalize (19) to the following directional generator:

$$\frac{1}{2\pi i}\oint_z dw\omega(w)J(w) \tag{20}$$

where the analytic function $\omega(w) = \sum_a \omega^a(w)t^a$ is regarded as a direction and we define

$$\omega(w)J(w) := \sum_a \omega^a(w)J^a \tag{21}$$

Then since $W(z,z') \in SU(2)\otimes U(1)$, from the variational formula (20) for the loop algebra of the loop group of $SU(2)\otimes U(1)$ we have that the variation of $W(z,z')$ in the direction $\omega(w)$ is given by

$$W(z,z')\frac{1}{2\pi i}\oint_z dw\omega(w)J(w) \tag{22}$$

Now let us consider the quantum case which is based on the quantum gauge model in section 2. For this quantum case we shall choose a quantum generator J which is analogous to the J in (17). Let us consider the following correlation which is a functional integration:

$$\langle W(z,z')A(z)\rangle := \int dA_1^*dA_1dA_2^*dA_2dZ^*dZe^{-L}W(z,z')A(z) \tag{23}$$

where $A(z)$ denotes a field from the quantum gauge model (We first let z' be fixed as a parameter).

Let us derive a Ward identity by applying a gauge transformation on (23) as follows. Let (A_1, A_2, Z) be regarded as a coordinate system of the integral (23). Under a gauge transformation (regarded as a change of coordinate) this coordinate is changed to another coordinate denoted by (A_1', A_2', Z') and we have the following equality:

$$\begin{array}{rl} & \int dA_1'^*dA_1'dA_2'^*dA_2'dZ'^*dZ'e^{-L'}W'(z,z')A'(z) \\ = & \int dA_1^*dA_1dA_2^*dA_2dZ^*dZe^{-L}W(z,z')A(z) \end{array} \tag{24}$$

where $W'(z,z')$ denotes the Wilson line based on A_1' and A_2' and similarly $A'(z)$ denotes the field obtained from $A(z)$ with (A_1, A_2, Z) replaced by (A_1', A_2', Z').

Then by the gauge invariance property the differential

$$e^{-L}dA_1^*dA_1dA_2^*dA_2dZ^*dZ \tag{25}$$

is unchanged under a gauge transformation [8]. Thus we have

$$0 = \langle W'(z,z')A'(z)\rangle - \langle W(z,z')A(z)\rangle \tag{26}$$

where the correlation notation $\langle\rangle$ denotes the integral with respect to the differential (25).

We can now carry out a calculus of variation for the Ward identity. From the gauge transformation we have the formula $W'(z,z') = U(\omega(z)W(z,z')U^{-1}(\omega(z'))$ This gauge transformation gives a variation of $W(z,z')$ with the function ω as the variational direction ω in the variational formulas (20) and (22). Thus analogous to the variational formula (22) we have that an ansatz of the variation of $W(z,z')$ under this gauge transformation is given by

$$W(z,z')\frac{1}{2\pi i}\oint_z dw\omega(w)J(w) \tag{27}$$

where the generator J for this variation is to be specified. This J will be a quantum generator which generalizes the classical generator J in (22).

Thus under a gauge transformation from (26) we have the following variational equation:

$$0 = \langle W(z,z')[\delta_\omega A(z) + \frac{1}{2\pi i}\oint_z dw\omega(w)J(w)A(z)]\rangle \tag{28}$$

where $\delta_\omega A(z)$ denotes the variation of the field $A(z)$ in the direction ω. From this equation an ansatz of J is that J satisfies the following equation:

$$W(z,z')[\delta_\omega A(z) + \frac{1}{2\pi i}\oint_z dw\omega(w)J(w)A(z)] = 0 \tag{29}$$

From this equation we have the following variational equation:

$$\delta_\omega A(z) = \frac{-1}{2\pi i}\oint_z dw\omega(w)J(w)A(z) \tag{30}$$

This completes the calculus of variation.

Let us now determine the generator J in (30). As analogous to the WZW model in conformal field theory [16] [14] let us consider a J given by

$$J(z) := -kW^{-1}(z,z')\partial_z W(z,z') \tag{31}$$

where we set $z' = z$ after the differentiation with respect to z; $k > 0$ is a constant which is fixed when the J is determined to be of the form (31) and the minus sign is chosen by convention. In the WZW model [16][14] the J of the form (31) is the generator of the chiral symmetry of the WZW model. We can write the J in (31) in the following form:

$$J(w) = \sum_a J^a(w) = \sum_a j^a(w)t^a = \sum_a \sum_{n=-\infty}^{\infty} J_n^a (w-z)^{-n-1} \tag{32}$$

We see that the generators t^a of the gauge group appear in this form of J and this form is completely analogous to the classical J in (17). This shows that this J is a possible candidate for the generator J in (30).

Here let us consider a property of the gauge invariance of gauge field. Because of gauge invariance there is a freedom to choose a gauge for a gauge field and it is needed to fix a gauge for computation and for the gauge model to be well defined. [8].

Let us then consider again the J in (31) and the Wilson line $W(z,z')$. Since $W(z,z')$ is constructed with a gauge field we need to have a gauge fixing for the computations related to $W(z,z')$. Then since the J in (31) is constructed from $W(z,z')$ we have that in choosing this J as the generator J in (30) we have indirectly added a condition for the gauge fixing.

In this paper we shall always choose this gauge fixing condition. With this gauge fixing condition the quantum gauge model is then completed.

Now we want to show that this generator J in (31) can be uniquely solved (This also means that the gauge fixing condition has already fixed the gauge that we can carry out computation). From (10) and (31) we have that the variation $\delta_\omega J$ of the generator J in (31) is given by [14][16]:

$$\delta_\omega J = [J,\omega] - k\partial_z\omega \tag{33}$$

From (30) and (33) we have that J satisfies the following relation of current algebra [14][15][16]:

$$J^a(w)J^b(z) = \frac{k\delta_{ab}}{(w-z)^2} + \sum_c if_{abc}\frac{J^c(z)}{(w-z)} \tag{34}$$

where as a convention the regular term of the product $J^a(w)J^b(z)$ is omitted. Then by following [14][15][16] from (34) and (32) and we can show that the J_n^a in (32) satisfy the following Kac-Moody algebra:

$$[J_m^a, J_n^b] = if_{abc}J_{m+n}^c + km\delta_{ab}\delta_{m+n,0} \tag{35}$$

where k is usually called the central extension or the level of the Kac-Moody algebra.

Let us then consider the other side of the chiral symmetry. Similar to the J in (31) we define a generator J' by:

$$J'(z') = k\partial_{z'}W(z,z')W^{-1}(z,z') \tag{36}$$

where after differentiation with respect to z' we set $z = z'$. Let us then consider the following correlation:

$$\langle A(z')W(z,z')\rangle := \int dA_1^* dA_1 dA_2^* dA_2 dZ^* dZ A(z')W(z,z')e^{-L} \tag{37}$$

where z is fixed. By an approach similar to the above derivation of (30) we have the following variational equation:

$$\delta_{\omega'}A(z') = \frac{-1}{2\pi i}\oint_{z'} dw A(z')J'(w)\omega'(w) \tag{38}$$

where as a gauge fixing we choose the J' in (38) be the J' in (36). Then similar to (33) we also we have

$$\delta_{\omega'} J' = [J', \omega'] - k\partial_{z'}\omega' \tag{39}$$

Then from (38) and (39) we can derive the current algebra and the Kac-Moody algebra for J' which are of the same form of (34) and (35). From this we have $J' = J$.

Now with the above current operator J and the formula (30) we can follow the usual approach in conformal field theory to derive the Knizhnik-Zamolodchikov equation for the product of primary fields in a conformal field theory [14][15][17]. In our case here we derive the Knizhnik-Zamolodchikov equation for the product of n Wilson lines $W(z, z')$. Here from the two sides of $W(z, z')$ we can derive two Knizhnik-Zamolodchikov equations which are inversely dual to each other.

We have the following Knizhnik-Zamolodchikov equation [14] [15][17]:

$$\partial_{z_i} W(z_1, z_1') \cdots W(z_n, z_n') = \frac{-1}{k+g}\sum_{j\neq i}^{n} \frac{\sum_a t_i^a \otimes t_j^a}{z_i - z_j} W(z_1, z_1') \cdots W(z_n, z_n') \tag{40}$$

for $i = 1, ..., n$ where g denotes the dual Coxeter number for $SU(2)$ and we have $g = 2e^2$.

We also have the following Knizhnik-Zamolodchikov equation with repect to the z_i' variables which is dual to (40):

$$\partial_{z_i'} W(z_1, z_1') \cdots W(z_n, z_n') = \frac{-1}{k+g}\sum_{j\neq i}^{n} W(z_1, z_1') \cdots W(z_n, z_n') \frac{\sum_a t_i^a \otimes t_j^a}{z_j' - z_i'} \tag{41}$$

for $i = 1, ..., n$.

6 Winding Number of Dirac-Wilson Loop as Quantization

In this section we solve the Dirac-Wilson Loop in a form with a winding property. We show that there are discrete winding numbers come out from the Dirac-Wilson loop. When modeling photon by the abelian Dirac-Wilson loop these discrete winding numbers will be regarded as the quantization of energy of the Planck's formula of radiation.

Let us consider the following product of two Wilson lines:

$$G(z_1, z_2, z_3, z_4) := W(z_1, z_2)W(z_3, z_4) \tag{42}$$

where the two Wilson lines $W(z_1, z_2)$ and $W(z_3, z_4)$ represent two pieces of curves starting at z_1 and z_3 and ending at z_2 and z_4 respectively.

We have that this product G satisfies the KZ equation for the variables z_1, z_3 and satisfies the dual KZ equation for the variables z_2 and z_4. Let us first consider the case that the gauge group is $SU(2)$. Then by solving the two-variables-KZ equation in (40) we have that a form of G is given by [18][19][20]

$$e^{-t\log[\pm(z_1 - z_3)]} C_1 \tag{43}$$

where $t := \frac{1}{k+g}\sum_a t^a \otimes t^a$ is a Casimir operator of $SU(2)$ and C_1 denotes a constant matrix which is independent of the variable $z_1 - z_3$. We see that G is a multivalued analytic function where the determination of the $\pm$ sign depended on the choice of the branch.

Similarly by solving the dual two-variable-KZ equation in (41) we have that G is of the form

$$C_2 e^{t\log[\pm(z_4 - z_2)]} \tag{44}$$

where C_2 denotes a constant matrix which is independent of the variable $z_4 - z_2$.

From (43), (44) and we let $C_1 = Ae^{t\log[\pm(z_4 - z_2)]}$, $C_2 = e^{-t\log[\pm(z_1 - z_3)]}A$ where A is a constant matrix as an initial condition we have that G is given by

$$G(z_1, z_2, z_3, z_4) = e^{-t\log[\pm(z_1 - z_3)]} A e^{t\log[\pm(z_4 - z_2)]} \tag{45}$$

Let us set $z_2 = z_3$. In this case the degree of freedom of $W(z_1, z_2)W(z_3, z_4)$ is reduced and we have $W(z_1, z_4) = W(z_1, z_2)W(z_2, z_4)$. Then since t is a Casimir operator for $SU(2)$ and A is an initial operator for $W(z_1, z_4)$ we have that $\Phi = e^{-t\log[\pm(z_1-z_3)]}$ and $\Psi = e^{t\log[\pm(z_4-z_2)]}$ as matrix acted on A commute with A since Φ and Ψ are exponentials of t. Thus we have

$$W(z_1, z_4) = W(z_1, z_2)W(z_2, z_4) = e^{-t\log[\pm(z_1-z_2)]}Ae^{t\log[\pm(z_4-z_2)]} = e^{-t\log[\pm(z_1-z_2)]}e^{t\log[\pm(z_4-z_2)]}A \quad (46)$$

Now let $z_1 = z_4$. In this case we have a closed loop. Now in (46) we have that $e^{-t\log[\pm(z_1-z_2)]}$ and $e^{t\log[\pm(z_1-z_2)]}$ cancel each other and from the multivalued property of the log function we have

$$W(z_1, z_1) = RA \quad (47)$$

where $R := e^{-in\pi t}$ for $n = 0, \pm1, \pm2, ...$ is the monodromy of the KZ equation for $SU(2)$ where the integer n is as a winding number. In choosing an integer n we have chosen a branch of R.

Similarly when the gauge group is $SU(2) \otimes U(1)$ we have

$$W(z_1, z_1) = R_{U(1)}R_{SU(2)}A \quad (48)$$

where $R_{SU(2)} := R = e^{-in\pi t}$ for $n = 0, \pm1, \pm2, ...$ is the monodromy of KZ equation for $SU(2)$ and $R_{U(1)}$ denotes the monodromy of the KZ equation for $U(1)$. We have $R_{U(1)} = e^{in\frac{e^2\pi}{k+g}}$ for $n = 0, \pm1, \pm2, ...$ where the e of e^2 denotes the electric charge. We shall show that the winding number n gives the quantization property of photon.

Now we have that the Dirac-Wilson loop $W(z_1, z_1)$ corresponds to a closed curve in the complex plane with starting and ending point z_1. Let this Dirac-Wilson loop $W(z_1, z_1)$ represents the unknot. Then we let the gauge group be just the $U(1)$ group. In this case we show in the following section that the Dirac-Wilson loop $W(z_1, z_1)$ is a model of the photon.

7 Photon as the Magnetic Monopole

We see that the Dirac-Wilson loop is an exactly solvable nonlinear observable. Thus we may regard it as a quantum soliton of the above gauge model. In particular for the abelian gauge model with $U(1)$ as gauge group we regard the Dirac-Wilson loop as a quantum soliton of the electromagnetic field. We now want to show that this soliton has all the properties of photon and thus we may identify it with the photon. First we see that it has discrete energy levels of light-quantum given by

$$nh\nu := n\frac{\pi e^2}{k+g}, \qquad n = 0, 1, 2, 3, ... \quad (49)$$

where h is the Planck's constant; ν denotes a frequency and the constant k is determined from this formula. This formula is from the monodromy $R_{U(1)}$ for the abelian gauge model. We see that the Planck's constant h comes out from this winding property of the Dirac-Wilson loop. Then since this Dirac-Wilson loop is a loop we have that it has the polarization property of light by the right hand rule along the loop and this polarization can also be regarded as the spin of photon. Now since this loop is a quantum soliton which behaves as a particle we have that this loop is a basic particle of the above abelian gauge model where the abelian gauge property is considered as the fundamental property of electromagnetic field. This shows that the Dirac-Wilson loop has properties of photon. We shall later show that from this loop model of photon we can describle the absorption and emission of photon by an electron. This property of absorption and emission is considered as a basic principle of the light-quantum hypothesis of Einstein [1]. From these properties of the Dirac-Wilson loop we may identify it with the photon.

On the other hand from Dirac's analysis of the magnetic monopole [4] we have that the property of magnetic monopole comes from a closed line integral of vector potential of the electromagnetic field which is similar to the Dirac-Wilson loop. Now from this Dirac-Wilson loop we can define the magnetic charge q which is given by

$$eq := e\frac{e\pi n}{k+g}, \qquad n = 0, 1, 2, 3, ... \quad (50)$$

where e denotes the electric charge. This shows that the Dirac-Wilson loop gives the property of magnetic monopole. Since this loop is a quantum soliton which behaves as a particle we have that this Dirac-Wilson loop may be identified with the magnetic monopole. Thus we have that photon may be identified with the magnetic monopole.

With this identification we have the following interesting conclusion: Both the energy quantization of electromagnetic field and the charge quantization property come from the same property of photon. Indeed we have

$$nh\nu = n\frac{e^2\pi}{k+g} = e\frac{e\pi n}{k+g} = eq, \qquad n = 0, 1, 2, 3, ... \tag{51}$$

This formula shows that the energy quantization and the charge quantization are just the same quantization and is a property of the photon when photon is modeled by the Dirac-Wilson loop and identified with the magnetic monopole.

8 Nonlinear Model of Electron

In this section let us also give a loop model to the electron. This loop model of electron is based on the above loop model of the photon. From the loop model of photon we also construct an observable which gives mass to the electron and is thus a mass machanism for the electron.

Let W(z,z) denote a Dirac-Wilson loop which represents a photon. Let Z denotes the complex variable for electron in (2). We then consider the following observable:

$$W(z,z)Z \tag{52}$$

Since $W(z,z)$ is solvable we have that this observable is also solvable where in solving $W(z,z)$ the variable Z is fixed. We let this observable be identified with the electron. Then we consider the following observable:

$$Z^*W(z,z)Z \tag{53}$$

This observable is with a scalar factor Z^*Z where Z^* denotes the complex conjugate of Z and we regard it as the mass mechanism of the electron (52). For this observable we model the energy levels of $W(z,z)$ as the mass levels of electron and the mass m of electron is the lowest nonzero energy level $h\nu$ of $W(z,z)$ and is given by:

$$mc^2 = h\nu \tag{54}$$

where c denotes the constant of the speed of light. From this model of the mass mechanism of electron we have that electron is with mass m while photon is with zero mass because there does not have such a mass mechanism $Z^*W(z,z)Z$ for the photon. From this definition of mass we have the following formula relating the charge e of electron, the magnetic charge q_{min} of magnetic monopole and the mass m of electron:

$$mc^2 = eq_{min} = h\nu \tag{55}$$

By using the nonlinear model $W(z,z)Z$ to represent an electron we can then describe the absorption and emission of a photon by an electron where photon is as a parcel of energy described by the loop $W(z,z)$, as follows. Let $W_1(z_1,z_1)Z$ represents an electron and let $W_2(z',z')$ represents a photon. Then the observable $(W_1(z_1,z_1) + W_2(z_2,z_2))Z(z)$ represents that the photon $W_2(z',z')$ is absorbed by the electron $W_1(z_1,z_1)Z$. This observable $(W_1(z_1,z_1)+W_2(z_2,z_2))Z$ again represents an electron. This then gives a model of the absorption of a photon by an electron. Similarly we may model the emission of a photon by an electron.

This property of absorption and emission is as a principle of the light-quantum hypothesis of Einstein [1]. Let us quote the following paragraph from [1]:

..., First, the light-quantum was conceived of as a parcel of energy as far as the properties of pure radiation (no coupling to matter) are concerned. Second, Einstein made the assumption–he call it the heuristic principle–that also in its coupling to matter (that is, in emission and absorption), light is created or annihilated in similar discrete parcels of energy. That, I believe, was Einstein's one revolutionary contribution to physics. It upset all existing ideas about the interaction between light and matter. ....

9 Statistics of Photons and Electrons

Let us consider more on the nonlinear model of electron $W(z,z)Z$ which gives a relation between photons and electrons where photons is modeled by $W(z,z)$. We want to show that from this nonlinear model we may also derive the required statistics of photons and electrons that photons obey the Bose-Einstein statistics and electrons obey the Fermi-Dirac statistics. We have that $W(z,z)$ is as an operator acting on Z which represents an electron while the nonlinear model of electron $W(z,z)Z$ gives a more complete description of electron that it describes that a photon $W(z,z)$ is absorbed by an electron and the two particles again form an electron. There may be many photons $W_n(z_n, z_n), n = 1, 2, 3, ...$ absorbed by an electron which gives an electron $\sum_n W_n(z_n, z_n)Z$. This formula shows that identical (but different) photons can appear identically and it shows that photons obey the Bose-Einstein statistics. From the polarization of the Dirac-Wilson loop $W(z,z)$ we may assign spin 1 to a photon represented by $W(z,z)$.

Let us then consider statisics of electrons. We have that the observable $Z^*W(z,z)Z$ gives mass to the electron $W(z,z)Z$ and thus this observable is as a scalar and thus is assigned with spin 0. As the observable $W(z,z)Z$ is between $W(z,z)$ and $Z^*W(z,z)Z$ which are with spin 1 and 0 respectively we thus assign spin $\frac{1}{2}$ to the observable $W(z,z)Z$ and thus electron represented by this observable $W(z,z)Z$ is with spin $\frac{1}{2}$.

Then let Z_1 and Z_2 be two independent complex variables for two electrons. Let $W(z,z)$ represents a photon. Then the operator $W(z,z)(Z_1 + Z_2)$ means that two electrons are in the same state that they are acted by the same photon. However this operator means that a photon $W(z,z)$ is absorbed by two distinct electrons and this is impossible since a photon can only be absorbed by one electron. Thus this operator $W(z,z)(Z_1 + Z_2)$ cannot be observable and this means that electrons obey Fermi-Dirac statistics.

Thus we have that this nonlinear loop model of photon and electron give the required statistics of photons and electrons.

10 Conclusion

In this paper a quantum loop model of photon is set up. We show that this loop model is exactly solvable and thus may be considered as a quantum soliton. We show that this nonlinear model of photon has properties of photon and magnetic monopole and thus photon may be identified with the magnetic monopole. From the discrete winding numbers of this loop model we can derive the quantization property of energy for the Planck's formula and the quantization property of electric charge. We show that energy quantization and charge quantization are just the same quantization. On the other hand from the nonlinear model of photon a nonlinear loop model of electron is set up. This model of electron has a mass machanism which generates mass to the electron.

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